6. EXERCISE SHEET, RETURN DATE JUNE 18TH/19TH

Important!

We now have group accounts in the unixpool. The individual accounts will be disabled during this week. Until then, please copy everything from every individual account to your group account. I suggest making 4 directories with the names of your group members. You can use the linux command *scp* to copy your files. See wikipedia for details. The group accounts have numbers 22x, where x is your group number.

Exercise 1

In this exercise we want to calculate geodesics on the paraboloid. The paraboloid is the manifold

$$\mathcal{M} := \{(x, y, z) : z = x^2 + y^2\}$$

A geodesic is a curve in a manifold, whose curvature vector lies in the normal space to the manifold: $\gamma : \mathbb{R} \to \mathcal{M}$ with $\langle \ddot{\gamma}(t), v\rangle = 0\,\forall v \in T_{\gamma(t)}\mathcal{M}$. This implies automatically that the speed of a geodesic stays constant. We can fix a geodesic by supplying an initial position $\gamma(0)$ and an initial velocity $\dot{\gamma}(0) \in T_{\gamma(0)}\mathcal{M}$.

The tangent space on a point $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{M}$ has the basis $\{\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \\ 2x^2 + 2y^2 \end{pmatrix}\}$, which is not unique in the origin.

Basic linear algebra yields the equivalent definition of this geodesic by the following differential equation:

$$\dot{a} = -x \cdot \frac{4(a^2 + b^2)}{1 + 4(x^2 + y^2)}$$

$$\dot{b} = -y \cdot \frac{4(a^2 + b^2)}{1 + 4(x^2 + y^2)}$$

$$\dot{c} = \frac{2(a^2 + b^2)}{1 + 4(x^2 + y^2)}$$

$$\dot{x} = a,\ \dot{y} = b,\ \dot{z} = c$$

Now check if this looks like a geodesic on the paraboloid by using some numerical integration technique, for example the explicit Euler method. If you wish, you may copy functions from surfaceplot.c provided on the website for making a 3D-plot of the paraboloid and your geodesic.

Exercise 2

Try whether you can also calculate the geodesic more directly. For example by the following method:

(1) move $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ along $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

(2) Find intersection of the normal space with the paraboloid.

(3) Somehow update $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

This exercise has an open end. You are not required to create a method that works. However you should be able to plot the solution curve of your method for different step sizes in the same plot with the correct curve from exercise 1, such that we can compare. Try to find a method converging to the exact solution though.

Exercise 3

Implement the gradient descent and the conjugate gradient method which you can find on wikipedia. The method should be able to solve the linear system $Ax = b$, where A is a symmetric positive definite matrix.

Exercise 4 (optional)

Implement an implicit scheme for the problem in exercise 3 of the 4th homework sheet.

Exercise 5 (optional)

Use Taylor expansion to derive finite difference quotients for the first derivative with an order of consistency of 1, 2 and 3.

Exercise 6 (optional)

Extend your spaceship program to the situation in which the spaceship has the mass of the earth, or:

Extend your spaceship program to the situation in which there is a moon. Can you make the space ship orbit in a figure 8 around the two bodies. Remember that step size control is essential for attaining accurate results. As this three-body problem produces an ODE in 12 dimensions. Consider implementing the Heun with step size control for arbitrarily dimensioned ODE systems. Maybe you can even simulate the n-body problem for large n? Or extend your simulation to three dimensions?